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Integral Representation of Normalized Weak Markov Systems*

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A necessary and sufficient condition for the existence of an integral representation of weak Markov systems is given. This theorem generalizes results of Zalik and Zielke. The proof is based on the relative differentiation method of weak Markov systems introduced by Zielke, and on new alternation and oscillation properties of weak M^+ systems, which may be of some independent interest. © 1992 Academic Press, Inc.

TERMINOLOGY AND RESULTS

For a nonempty subset of the real line, $A \subset \mathbb{R}$, let us denote its convex hull by $K(A)$. Let $M \subset \mathbb{R}$ with card $M \geqslant n+2$, $c \in M$, and $h : M \to \mathbb{R}$ strictly increasing with $h(c) = c$. Moreover, let $J := K(h(M))$, and let $w_1, ..., w_n \in C(J)$ be increasing functions with $w_j(c) = 0$ for every $j \in \{1, ..., n\}$.

Define, for $x \in M$,

$$\begin{aligned} g_1(x) &= \int_c^{h(x)} dw_1(t_1) \\ g_2(x) &= \int_c^{h(x)} \int_c^{t_1} dw_2(t_2)\, dw_1(t_1) \\ &\vdots \\ g_n(x) &= \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_{n-1}} dw_n(t_n) \cdots dw_2(t_2)\, dw_1(t_1), \end{aligned} \tag{I}$$

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and for $t \in J$,

$$v_1(t) = \int_c^t dw_1(t_1)$$
$$\vdots \tag{I1}$$
$$v_n(t) = \int_c^t \int_c^{t_1} \cdots \int_c^{t_{n-1}} dw_n(t_n) \cdots dw_2(t_2)\, dw_1(t_1),$$

$$u_2(t) = \int_c^t dw_2(t_2)$$
$$\vdots \tag{I2}$$
$$u_n(t) = \int_c^t \int_c^{t_2} \cdots \int_c^{t_{n-1}} dw_n(t_n) \cdots dw_3(t_3)\, dw_2(t_2).$$

Moreover, let $F(M) := \{f\colon M \to \mathbb{R}\}$, $f_0, ..., f_n \in F(M)$, $\Delta_k(M) := \{x \in M^k : x_1 < \cdots < x_k\}$ for $k \in \mathbb{N}$, and $U_i := \operatorname{lin}\{f_0, ..., f_i\}$ for $i \in \{0, ..., n\}$.

Provided that $f_0, ..., f_n \in F(M)$ are linearly independent and $\det(f_i(t_j))_{0 \leqslant i,j \leqslant n}$ has a weakly constant sign for all $(t_0, ..., t_n) \in \Delta_{n+1}(M)$, $f_0, ..., f_n$ is called a weak Tchebycheff system on M; we say $f_0, ..., f_n \in F(M)$ is a weak T^+ system, if the sign is nonnegative. $f_0, ..., f_n \in F(M)$ is called a weak Markov system (weak M^+ system) on M, if $f_0, ..., f_j$ is a weak Tchebycheff system (weak T^+ system) for every $j \in \{0, ..., n\}$.

If, in addition, $f_0 \equiv 1$, a weak Markov system $f_0, ..., f_n$ (weak M^+ system) is called normalized.

A normalized weak Markov system $1, f_1, ..., f_n \in F(M)$ is called representable, if there are functions $1, g_1, ..., g_n \in F(M)$ defined by (I) with $\operatorname{lin}\{1, ..., g_j\} = \operatorname{lin}\{1, ..., f_j\}$ for every $j \in \{1, ..., n\}$.

DEFINITION. Let $f \in F(M)$. Points $(x_1, ..., x_k) \in \Delta_k(M)$ are called a strong alternation of length k of f, if there exists $\tau \in \{-1, 1\}$, such that

$$\tau(-1)^{k-i} f(x_i) > 0 \qquad \text{for} \quad i = 1, ..., k.$$

A strong alternation is called positively oriented, if and only if $\tau = 1$.

The following lemma is [10, Lemma 4.1]

LEMMA 1. *Let $f_0, ..., f_n \in F(M)$ be linearly independent. Then the following two statements are equivalent*:

(a) *$f_0, ..., f_n$ is a weak Tchebycheff system*;

(b) *No $f \in U_n$ has a strong alternation of length $n+2$.*

Subsequently, we shall derive some new alternation and oscillation properties of weak M^+ systems (Lemmas 2 and 4) and use them to obtain new properties of representable weak Markov systems.

LEMMA 2. *Let* $f_0, ..., f_n \in F(M)$ *be a weak Markov system. Then the following statements are equivalent*:

(a) $f_0, ..., f_n$ *is a weak* M^+ *systems*;

(b) $f_0 \geq 0$ *and for each function* $f = \alpha f_n + g$, $g \in U_{n-1}$, $\alpha \neq 0$, *with a strong alternation of length* $n+1$ *in* M, *the alternation is positively oriented, if and only if* $\alpha > 0$.

DEFINITION. Let $k \geq 2$. An $f \in F(M)$ has a strong oscillation of length k if there exists $(x_1, ..., x_k) \in \Delta_k(M)$ and $\tau \in \{-1, 1\}$, such that

$$\tau(-1)^{k-i}(f(x_i) - f(x_{i-1})) > 0 \qquad \text{for} \quad i = 2, ..., k.$$

The strong oscillation is called positively oriented, if and only if $\tau = 1$.

The following lemma was developed in [11, 14]. In [12] an elementary proof, without use of the Gaußkernel approximation of weak Markov systems by smooth Markov systems, was given.

LEMMA 3. *Let* $1, f_1, ..., f_n \in F(M)$ *be a normalized weak Markov system. Then no* $f \in U_n$ *has a strong oscillation of length* $n+2$.

LEMMA 4. *Let* $1, f_1, ..., f_n \in F(M)$ *be a normalized weak* M^+ *system. If the function* $f \in U_n$ *with* $f = \alpha f_n + g$, $g \in U_{n-1}$, $\alpha \neq 0$, *has a strong oscillation of length* $n+1$, *then the oscillation is positively oriented, if and only if* $\alpha > 0$.

DEFINITION. A normalized weak Markov system $1, f_1, ..., f_n \in F(M)$ is called weakly nondegenerate, if for every a, $b \in M$ and for every $j \in \{0, ..., n-1\}$

$$f_{j|(a,b)\cap M} \in U_{j-1} \qquad \Rightarrow \qquad f_{j+1|(a,b)\cap M} \in U_{j-1}.$$

Our definition of weak nondegeneracy is different from the definition introduced by Zalik in [9].

We will prove

THEOREM 1. *Every representable weak Markov system* $1, g_1, ..., g_n \in F(M)$ *is weakly nondegenerate.*

DEFINITION. A normalized weak Markov system $1, f_1, ..., f_n \in F(M)$ has Property (E), if the following conditions are satisfied:

(E1) There exists a normalized weak M^+ system $1, g_1, ..., g_n \in F(M)$ such that $\text{lin}\{1, ..., g_j\} = \text{lin}\{1, ..., f_j\}$ for every $j \in \{1, ..., n\}$.

(E2) For every point $c \in K(M)$ with $\dim U_{n|[c,\infty)\cap M} = n+1$, there exists $1, u_1, ..., u_n \in F(M)$ with $u_j - g_j \in \text{lin}\{1, ..., g_{j-1}\}$ for every $j \in \{1, ..., n\}$, such that for any ordered subsequence $(k(l))_{l=0}^m$ of $\{0, ..., n\}$ the functions $u_{k(0)}, ..., u_{k(m)}$ form a weak M^+ system on $[c, \infty) \cap M$.

(E3) For every point $c \in K(M)$ with $\dim U_{n|(-\infty,c]\cap M} = n+1$, there exists $1, v_1, ..., v_n \in F(M)$ with $v_j - g_j \in \text{lin}\{1, ..., g_{j-1}\}$ for every $j \in \{1, ..., n\}$, such that for any ordered subsequence $(k(l))_{l=0}^m$ of $\{0, ..., n\}$ the functions $(-1)^{k(0)-0} v_{k(0)}, ..., (-1)^{k(m)-m} v_{k(m)}$ form a weak M^+ system on $(-\infty, c] \cap M$.

In [9] Zalik introduced Property (E) for weak Markov systems, and he gave an integral representation for weak normalized Markov systems with the conditions of Property (E) and the following Condition (I) (see Theorem 3 in [9]).

Condition (I). *For every real number* c, *the weak Markov system is linearly independent on at least one of the sets* $(c, \infty) \cap M$ *and* $(-\infty, c) \cap M$.

A representable weak Markov system does not fulfill Condition (I) in general as the following example shows:

Let $M = \{-1, 0, 1\}$ and let the functions $f_0, f_1, f_2 \in F(M)$ defined by $f_i(t) = t^i$, $i \in \{0, 1, 2\}$.

Zielke has shown in [11] that every nondegenerate normalized weak Markov system is representable. A weak Markov system is called nondegenerate, if for every $c \in M$ the functions are linearly independent on both of the sets $(c, \infty) \cap M$ and $(-\infty, c) \cap M$.

Our main result is

THEOREM 2. *A normalized weak Markov system* $1, f_1, ..., f_n \in F(M)$ *is representable, if and only if it has Property* (E).

PROOFS OF THE RESULTS

Proof of Lemma 2. We proceed by induction over n.

(a) $\Rightarrow$ (b): For $n = 0$ the statement is trivial.

$n-1 \Rightarrow n$: Let $(t_0, ..., t_n) \in \Delta_{n+1}(M)$ be a negatively oriented alternation of length $n+1$ of $f = \alpha f_n + g$, $\alpha > 0$ and $g \in U_{n-1}$. An easy calculation shows

$$\det\begin{pmatrix} f_0 \cdots f_n \\ t_0 \cdots t_n \end{pmatrix} = \frac{1}{\alpha} \sum_{j=0}^{n} f(t_j)(-1)^{n-j} \det\begin{pmatrix} f_0 \cdots & & \cdots f_{n-1} \\ t_0 \cdots t_{j-1} & t_{j+1} \cdots & t_n \end{pmatrix} \leqslant 0.$$

Proceeding as in Lemma 4.1(b) ⇒ (a) Subcase 2 in [10], we get

$$\dim U_{n-1|\{t_0, \ldots, t_n\}} = n.$$

So there is $j_0 \in \{0, \ldots, n\}$ with

$$\det\begin{pmatrix} f_0 \cdots & & \cdots f_{n-1} \\ t_0 \cdots t_{j_0-1} & t_{j_0+1} \cdots & t_n \end{pmatrix} > 0.$$

Thus

$$\det\begin{pmatrix} f_0 \cdots f_n \\ t_0 \cdots t_n \end{pmatrix} < 0,$$

in contradiction to the fact that $f_0, \ldots, f_n$ is a weak M^+ system.

If $\alpha < 0$ and $(t_0, \ldots, t_n) \in \Delta_{n+1}(M)$ is positively oriented, then the statement follows completely analogously.

(b) ⇒ (a): The case $n = 0$ is trivial.

$n-1 \Rightarrow n$: By induction hypothesis $f_0, \ldots, f_{n-1}$ is a weak M^+ system. Suppose there exists $(x_0, \ldots, x_n) \in \Delta_{n+1}(M)$ such that

$$\det\begin{pmatrix} f_0 \cdots f_n \\ x_0 \cdots x_n \end{pmatrix} < 0.$$

Thus,

$$\det\begin{pmatrix} f_0 \cdots f_n \\ t_0 \cdots t_n \end{pmatrix} \leqslant 0$$

for every $(t_0, \ldots, t_n) \in \Delta_{n+1}(M)$.

Since $\dim U_{n|\{x_0, \ldots, x_n\}} = n+1$, there is exactly one $f \in U_n$ with

$$f(x_j) = (-1)^{n-j}, \qquad j = 0, \ldots, n.$$

Then we have $f = \alpha f_n + g$, $g \in U_{n-1}$, $\alpha > 0$, and

$$0 > \det\begin{pmatrix} f_0 \cdots f_n \\ x_0 \cdots x_n \end{pmatrix} = \frac{1}{\alpha} \sum_{j=0}^{n} f(x_j)(-1)^{n-j} \det\begin{pmatrix} f_0 \cdots & & \cdots f_{n-1} \\ x_0 \cdots x_{j-1} & x_{j+1} \cdots & x_n \end{pmatrix}$$

$$= \frac{1}{\alpha} \sum_{j=0}^{n} \det\begin{pmatrix} f_0 \cdots & & \cdots f_{n-1} \\ x_0 \cdots x_{j-1} & x_{j+1} \cdots & x_n \end{pmatrix}.$$

By the induction hypothesis

$$\det\begin{pmatrix} f_0 \cdots & & \cdots f_{n-1} \\ x_0 \cdots x_{j-1} & x_{j+1} \cdots & x_n \end{pmatrix} \geqslant 0, \qquad j = 0, \ldots, n,$$

in contradiction to

$$\det\begin{pmatrix} f_0 \cdots f_n \\ x_0 \cdots x_n \end{pmatrix} < 0. \qquad \text{Q.E.D.}$$

Proof of Lemma 4. Let $1, f_1, ..., f_n$ be a normalized weak M^+ system. If $n=1$, then the statement is trivial.

$n-1 \Rightarrow n$: Let $f \in U_n$ with $f = \alpha f_n + g$, $g \in U_{n-1}$, $\alpha > 0$, and let $(t_0, ..., t_n) \in \Delta_{n+1}(M)$ be a negatively oriented oscillation of length $n+1$, i.e.,

$$(-1)^{n-j}(f(t_j) - f(t_{j-1})) < 0, \qquad j = 1, ..., n.$$

Proceeding as in the proof of Lemma 4.1(b) $\Rightarrow$ (a) Subcase 2 in [10], we may assume that the restriction of U_{n-1} to $\{t_0, ..., t_n\} \subset M$ is a vector space of dimension n.

Case 1. $\dim U_{n-1 | \{t_0, ..., t_{n-1}\}} = n$.
For each $\varepsilon > 0$ there is a function $h_\varepsilon \in U_{n-1}$ such that

$$h_\varepsilon(t_j) = f(t_j) + \varepsilon(-1)^{n-j}, \qquad j = 0, ..., n-1.$$

Now fix an ε with $0 < \varepsilon < \frac{1}{2}\max\{|f(t_j) - f(t_{j-1})| \mid j = 1, ..., n\}$. By the induction hypothesis it follows that $h_\varepsilon(t_n) \geqslant h_\varepsilon(t_{n-1})$. Taking into consideration that

$$f(t_n) - h_\varepsilon(t_n) \leqslant f(t_n) - f(t_{n-1}) + \varepsilon < 0$$

and $(f - h_\varepsilon)(t_j) = \varepsilon(-1)^{n-1-j}$ for each $j \in \{0, ..., n-1\}$, we see that $(f - h_\varepsilon) \in U_n$ has a negatively oriented strong alternation of length $n+1$ in $(t_0, ..., t_n) \in \Delta_{n+1}(M)$, in contradiction to Lemma 2.

Case 2. $\dim U_{n-1 | \{t_0, ..., t_{n-1}\}} = n-1$.
We distinguish the following two subcases:

Subcase 2a. $\dim U_{n-1 | \{t_1, ..., t_n\}} = n$.
For every $\varepsilon > 0$ there is $h_\varepsilon \in U_{n-1}$ with

$$h_\varepsilon(t_j) = f(t_j) + \varepsilon(-1)^{n-j}, \qquad j = 1, ..., n.$$

Now, let us fix $\varepsilon > 0$ sufficiently small. By the induction hypothesis we have $(-1)^{n-1} h_\varepsilon(t_0) \leqslant (-1)^{n-1} h_\varepsilon(t_1)$, and

$$\begin{aligned}(-1)^n (f - h_\varepsilon)(t_0) &= (-1)^n f(t_0) + (-1)^{n-1} h_\varepsilon(t_0) \\ &\leqslant (-1)^n f(t_0) + (-1)^{n-1} h_\varepsilon(t_1) \\ &= (-1)^{n-1}(f(t_1) - f(t_0)) + \varepsilon.\end{aligned}$$

But then $(f-h_\varepsilon)\in U_n$ has a negatively oriented alternation in $(t_0, ..., t_n)\in \Delta_{n+1}(M)$, in contradiction to Lemma 2.

Subcase 2b. $\dim U_{n-1|\{t_0,...,t_{n-1}\}}=\dim U_{n-1|\{t_1,...,t_n\}}=n-1$.

For $h\in F(M)$, let us denote by $\hat{h}$ the restriction of h to $\{t_1, ..., t_n\}$. Since $\hat{f}_0, ..., \hat{f}_{n-1}$ are linearly dependent, there is a minimal $j\in\{0, ..., n-1\}$ with $\hat{f}_j\in\{\hat{f}_0, ..., \hat{f}_{j-1}\}$, say

$$\hat{f}_j=\sum_{i=0}^{j-1}\alpha_i\hat{f}_i, \qquad \alpha_i\in\mathbb{R}.$$

Then, proceeding analogously to the proof of Lemma 1 Subcase 1a in [12] we get that $\hat{f}_0, ..., \hat{f}_{j-1}, \hat{f}_{j+1}, ..., \hat{f}_n$ is a weak M^+ system. Now, applying the induction hypothesis, the strong oscillation $(t_1, ..., t_n)$ of $\hat{f}$ is positively oriented, and we arrive at a contradiction.

The proof for $\alpha<0$, and $(t_0, ..., t_n)\in\Delta_{n+1}(M)$ a positively oriented oscillation, is completely analogous. Q.E.D.

Following the argument used in the proof of Lemma 13.2 in [10] one gets:

LEMMA 5. *Let* $1, g_1, ..., g_n\in F(M)$ *be defined by* (I). *Then no* $g\in\operatorname{lin}\{1, g_1, ..., g_n\}$ *has a strong alternation of length* $n+2$.

For the proof of Theorem 1, the following two lemmas are essential.

LEMMA 6. *Let* $v_1, ..., v_n$ *be defined by* (I1), $k\in\{1, ..., n\}$, $[\alpha, \beta]\subset J$, *and* $v_{k|[\alpha,\beta]}\in\operatorname{lin}\{1, v_1, ..., v_{k-1}\}$. *Then there is a natural number* $l\in\mathbb{N}$ *and a partition* $\{x_0, ..., x_{l+1}\}$ *of* $[\alpha, \beta]$, *such that for every* $i\in\{0, ..., l\}$ *there is* $j_i\in\{1, ..., k\}$ *with* $w_{j_i}\equiv\text{const}$ *on* $[x_i, x_{i+1}]$.

Proof. Without loss of generality, we may assume $c=\alpha$. It is easy to see that replacing $c\in M$ by $\tilde{c}\in M$ the integral representation $1, v_1, ..., v_n$ leads to an integral representation $1, \tilde{v}_1, ..., \tilde{v}_n$, such that for every $i\in\{1, ..., n\}$: $\operatorname{lin}\{1, v_1, ..., v_i\}=\operatorname{lin}\{1, \tilde{v}_1, ..., \tilde{v}_i\}$ and $v_i-\tilde{v}_i\in\operatorname{lin}\{1, ..., v_{i-1}\}$.

We proceed by induction over n.

$n=1$: If $v_1\equiv 0$ on $[\alpha, \beta]$, then $w_1\equiv 0$ on $[\alpha, \beta]$.

$n-1\Rightarrow n$: Let $v_{n|[\alpha,\beta]}\in\operatorname{lin}\{1, v_1, ..., v_{n-1}\}$, then there is $v\in\operatorname{lin}\{1, v_1, ..., v_{n-1}\}$ with $v\equiv 0$ on $[\alpha, \beta]$ and $u\in\operatorname{lin}\{1, u_2, ..., u_{n-1}\}\setminus\{0\}$, such that $v(t)=\int_c^t u(t_1)\,dw_1(t_1)$, $t\in J$ where $u_2, ..., u_n$ are defined by (I2).

By Lemma 5 each alternation of u is of finite length, thus there is $x_1>c=\alpha$, such that either $u\equiv 0$ on (c, x_1) or $u(s)\neq 0$ for all $s\in(c, x_1)$; we may choose the interval (c, x_1) maximal.

Case 1. $u(x) \neq 0$ for every $x \in (c, x_1)$.

Without loss of generality, let $u(x) > 0$ on (c, x_1). Now, suppose $w_1(c) < w_1(t_0)$ for some $t_0 \in (c, x_1)$. Then there exists $\varepsilon > 0$, such that $w_1(c) < w_1(t)$ for every $t \in J$ with $|t - t_0| < \varepsilon$. But this implies $v(t) > 0$ for every $t \in [t_0, x_1]$, in contradiction to the fact that $w_1 \equiv \text{const}$ on (c, x_1).

Case 2. $u \equiv 0$ on (c, x_1).

Clearly, $u \equiv 0$ on $[c, x_1]$. By induction hypothesis there is a natural number l_1 and a partition $\{y_0, ..., y_{l_1+1}\}$ of $[c, x_1]$, such that for each $i_1 \in \{0, ..., l_1\}$ there exists $j_{i_1} \in \{2, ..., k\}$ with $w_{j_{i_1}} \equiv \text{const}$ on $[y_{i_1}, y_{i_1+1}]$.

We get $\int_c^{x_1} u(t_1)\, dw_1(t_1) = 0$ in both cases. Therefore, $v(t) = \int_{x_1}^t u(t_1)\, dw(t_1)$ on J. Since u has only finitely many separated zeros, repeated application of the argument used above yields a partition of $[\alpha, \beta]$. Q.E.D.

LEMMA 7. *Let $v_1, ..., v_n$ be defined by* (I1), $k \in \{1, ..., n\}$, $[\alpha, \beta] \subset J$, *and* $v_{k|[\alpha,\beta]} \in \text{lin}\{1, ..., v_{k-1}\}$. *Then for every* $p \in \{k+1, ..., n\}$ *there exists* $\alpha_p \in \mathbb{R}$, *such that*

$$v_{p|[\alpha,\beta]} = \alpha_p v_{k|[\alpha,\beta]}.$$

Proof. By Lemma 6 there exists $l \in \mathbb{N}$ and a partition $\{x_0, ..., x_l\}$ of $[\alpha, \beta]$, such that for every $i \in \{0, ..., l-1\}$ there is $j_i \in \{1, ..., k\}$ with $w_{j_i} \equiv \text{const}$ on $[x_i, x_{i+1}]$.

Without loss of generality we may assume:

(A) $\alpha = c$;

(B) for every $i \in \{0, ..., l-1\}$ and every $j \in \{j_i + 1, ..., k\}$, w_j is nonconstant on $[x_i, x_{i+1}]$.

If $l = 1$, we have $v_p \equiv 0$ on $[\alpha, \beta] = [x_0, x_1]$ for all $p \in \{j_0, ..., n\}$.

Now, let $l > 1$, and let $[x_i, x_{i+1}]$ and $[x_{i+1}, x_{i+2}]$ be arbitrarily fixed, so $w_{j_i} \equiv \text{const}$ on $[x_i, x_{i+1}]$ and $w_{j_{i+1}} \equiv \text{const}$ on $[x_{i+1}, x_{i+2}]$. For brevity let $q := j_i$ and $r := j_{i+1}$. Now, let us assume $q < r$. Then, for all $t \in [x_i, x_{i+1}]$ we have

$$\begin{aligned} v_k(t) &= \int_\alpha^t \cdots \int_\alpha^{t_{q-1}} \cdots \int_\alpha^{t_{k-1}} dw_k(t_k) \cdots dw_q(t_q) \cdots dw_1(t_1) \\ &= \int_\alpha^{x_i} \cdots \int_\alpha^{t_{k-1}} dw_k(t_k) \cdots dw_q(t_q) \cdot v_{q-1}(t) \end{aligned}$$

and

$$v_k(t) = \int_\alpha^{x_{i+1}} \cdots \int_\alpha^{t_{k-1}} dw_k(t_k) \cdots dw_r(t_r) \cdot v_{r-1}(t)$$

for all $t \in [x_{i+1}, x_{i+2}]$.

Since $r > q$, it follows that

$$v_{r-1}(t) = \int_\alpha^{x_i} \cdots \int_\alpha^{t_{r-2}} dw_{r-1}(t_{r-1}) \cdots dw_q(t_q) \cdot v_{q-1}(t)$$

for all $t \in [x_i, x_{i+1}]$, especially at the point x_{i+1}:

$$v_{r-1}(x_{i+1}) = \int_\alpha^{x_i} \cdots \int_\alpha^{t_{r-2}} dw_{r-1}(t_{r-1}) \cdots dw_q(t_q) \cdot v_{q-1}(x_{i+1}).$$

This implies

$$\begin{aligned} v_k(x_{i+1}) &= \int_\alpha^{x_i} \cdots \int_\alpha^{t_{k-1}} dw_k(t_k) \cdots dw_q(t_q) \cdot v_{q-1}(x_{i+1}) \\ &= \int_\alpha^{x_{i+1}} \cdots \int_\alpha^{t_{k-1}} dw_k(t_k) \cdots dw_r(t_r) \cdot v_{r-1}(x_{i+1}) \\ &= \int_\alpha^{x_{i+1}} \cdots \int_\alpha^{t_{k-1}} dw_k(t_k) \cdots dw_r(t_r) \\ &\quad \cdot \int_\alpha^{x_i} \cdots \int_\alpha^{t_{r-2}} dw_{r-1}(t_{r-1}) \cdots dw_q(t_q) \cdot v_{q-1}(x_{i+1}). \end{aligned}$$

We distinguish several cases and subcases:

Case 1.

$v_{q-1}(x_{i+1}) = 0$.
Then

$$v_{q-1}(t) = \int_\alpha^{t} \cdots \int_\alpha^{t_{q-2}} dw_{q-1}(t_{q-1}) \cdots dw_1(t_1) = 0$$

for all $t \in [\alpha, x_{i+1}]$, because v_{q-1} is increasing on $[\alpha, \infty) \cap J$ and $v_{q-1}(\alpha) = 0$.

This implies

$$\begin{aligned} 0 \leqslant v_p(t) &= \int_\alpha^{t} \cdots \int_\alpha^{t_{q-2}} \left(\int_\alpha^{t_{q-1}} \cdots \int_\alpha^{t_{p-1}} dw_p(t_p) \cdots dw_q(t_q) \right) \\ &\quad \times dw_{q-1}(t_{q-1}) \cdots dw_1(t_1) \\ &\leqslant \int_\alpha^{x_i} \cdots \int_\alpha^{t_{p-1}} dw_p(t_p) \cdots dw_q(t_q) \cdot v_{q-1}(t) = 0 \end{aligned}$$

for every $p > q-1$ and all $t \in [\alpha, x_{i+1}]$, so $v_p \equiv 0$ on $[\alpha, x_{i+1}]$ for $p \geqslant q$.

Case 2.

Now we assume

$$\underbrace{\int_\alpha^{x_i}\cdots\int_\alpha^{t_{k-1}} dw_k(t_k)\cdots dw_q(t_q)}_{=:C_1}=\underbrace{\int_\alpha^{x_{i+1}}\cdots\int_\alpha^{t_{k-1}} dw_k(t_k)\cdots dw_r(t_r)}_{=:C_2}$$
$$\cdot\underbrace{\int_\alpha^{x_i}\cdots\int_\alpha^{t_{r-2}} dw_{r-1}(t_{r-1})\cdots dw_q(t_q).}_{=:C_3}$$

For C_1 we have the following estimate:

$$\begin{aligned}0\leqslant C_1&=\int_\alpha^{x_i}\cdots\int_\alpha^{t_{r-2}}\left(\int_\alpha^{t_{r-1}}\cdots\int_\alpha^{t_{r-1}} dw_k(t_k)\cdots dw_r(t_r)\right)\\&\quad\times dw_{r-1}(t_{r-1})\cdots dw_q(t_q)\\&\leqslant\underbrace{\int_\alpha^{x_i}\cdots\int_\alpha^{t_{k-1}} dw_k(t_k)\cdots dw_r(t_r)}_{=:\tilde{C}_2}\\&\quad\cdot\underbrace{\int_\alpha^{x_i}\cdots\int_\alpha^{t_{r-2}} dw_{r-1}(t_{r-1})\cdots dw_q(t_q).}_{=C_3}\end{aligned}$$

Since $C_1=C_2\cdot C_3$, we have to deal with the following two subcases.

Subcase 2a. $C_3=0$.
Then for each $p>r-1$

$$\int_\alpha^{x_i}\cdots\int_\alpha^{t_{r-2}}\cdots\int_\alpha^{t_{p-1}} dw_p(t_p)\cdots dw_{r-1}(t_{r-1})\cdots dw_q(t_q)=0,$$

and therefore $v_p(x_i)=0$. This implies $v_p\equiv 0$ on $[\alpha, x_i]$, because $v_p(\alpha)=0$, and v_p is increasing on $[\alpha,\infty)\cap J$.

Subcase 2b. $\tilde{C}_2=C_2$.
If $k=r$, it follows that $w_r(x_{i+1})=w_r(x_i)$, and therefore $w_r\equiv\text{const}$ on $[x_i, x_{i+1}]$, in contradiction to assumption (B).

Now, let $k > r$. Then there exists $\zeta \in [x_i, x_{i+1}]$, such that

$$0 = \int_{x_i}^{x_{i+1}} \cdots \int_{\alpha}^{t_{k-1}} dw_k(t_k) \cdots dw_r(t_r)$$

$$= (w_r(x_{i+1}) - w_r(x_i)) \cdot \int_{\alpha}^{\zeta} \cdots \int_{\alpha}^{t_{k-1}} dw_k(t_k) \cdots dw_{r-1}(t_{r-1}).$$

This implies

$$\int_{\alpha}^{x_i} \cdots \int_{\alpha}^{t_{k-1}} dw_k(t_k) \cdots dw_{r-1}(t_{r-1}) = 0,$$

so $v_p \equiv 0$ on $[\alpha, x_i]$ for every $p \geqslant r$.

Summarizing the above considerations, we have in case $l > 1$:

For all intervals $[x_i, x_{i+1}]$ and $[x_{i+1}, x_{i+2}]$, $i \in \{0, ..., l-2\}$, with $j_i < j_{i+1}$ either

(a) $v_p \equiv 0$ on $[\alpha, x_{i+1}]$ for every $p \geqslant k$, and the sequence $(j_s)_{s=i+1}^{l-1}$ is strictly increasing, or

(b) $v_p \equiv 0$ on $[\alpha, x_i]$ for every $p \geqslant k$, the sequence $(j_s)_{s=i+1}^{l-1}$ is strictly increasing, and $C_1 = C_2 \cdot C_3$.

Now, consider the partition $\{x_{i+1}, ..., x_l\}$ of the subinterval $[x_{i+1}, \beta]$. For each interval $[x_s, x_{s+1}]$ with $s \geqslant 1$ we have

$$v_k(t) = \int_{\alpha}^{x_s} \cdots \int_{\alpha}^{t_{k-1}} dw_k(t_k) \cdots dw_{j_s}(t_{j_s}) \cdot v_{j_s-1}(t)$$

$$= \underbrace{\int_{\alpha}^{x_{i+1}} \cdots \int_{\alpha}^{t_{k-1}} dw_k(t_k) \cdots dw_{j_{i+1}}(t_{j_{i+1}})}_{=:\, C_{i+1,k}}$$

$$\cdot \underbrace{\prod_{v=i+2}^{s} \int_{\alpha}^{x_v} \cdots \int_{\alpha}^{t_{j_{v-1}}} dw_{j_{v-1}+1}(t_{j_{v-1}+1}) \cdots dw_{j_v}(t_{j_v}) \cdot v_{j_s-1}(t)}_{=:\, \gamma_s}$$

for all $t \in [x_s, x_{s+1}]$; if $s = i+1$, then let $\gamma_s = 1$.

Analogously we compute

$$v_p(t) = \underbrace{\int_{\alpha}^{x_{i+1}} \cdots \int_{\alpha}^{t_{p-1}} dw_p(t_p) \cdots dw_{j_{i+1}}(t_{j_{i+1}})}_{=:\, C_{i+1,p}} \cdot \gamma_s \cdot v_{j_s-1}(t)$$

for all $p \geqslant k$ on $[x_s, x_{s+1}]$.

Further distinctions are needed:

Case I. $C_{i+1,k}=0$.
Proceeding as in Case 1 one gets $C_{i+1,p}=0$ for every $p>k$. Therefore, $v_p\equiv 0$ on $[x_{i+1},\beta]$ for every $p\geqslant k$.
Besides, $C_{i+1,k}=0$ implies

$$\int_\alpha^t\cdots\int_\alpha^{t_{k-1}} dw_k(t_k)\cdots dw_{j_{i+1}}(t_{j_{i+1}})=0$$

for all $t\in[\alpha,x_{i+1}]$. So we have $v_p\equiv 0$ on $[\alpha,x_{i+1}]$ for every $p\geqslant k$.
Thus, $v_p\equiv 0$ on $[\alpha,\beta]$ for every $p\geqslant k$.

Case II. $C_{i+1,k}>0$.
For each interval $[x_s,x_{s+1}]$ with $s\geqslant i+1$ we have

$$\gamma_s\cdot v_{j_s-1}=\frac{v_k}{C_{i+1,k}},$$

so one gets

$$v_p\equiv\frac{C_{i+1,p}}{C_{i+1,k}}\cdot v_k$$

on the set $[\alpha,x_i]\cup[x_{i+1},\beta]$.
If $v_p\equiv 0$ on $[\alpha,x_{i+1}]$ for every $p\geqslant k$, obviously

$$v_p\equiv\frac{C_{i+1,p}}{C_{i+1,k}}\cdot v_k$$

on $[\alpha,\beta]$.
Now, let us assume $v_p\equiv 0$ on $[\alpha,x_i]$ and $C_1=C_2\cdot C_3$. Then, for all $t\in[x_i,x_{i+1}]$ and $p\geqslant k$ we have

$$v_p(t)=\underbrace{\int_\alpha^{x_i}\cdots\int_\alpha^{t_{p-1}} dw_p(t_p)\cdots dw_{j_i}(t_{j_i})}_{=:\,C_{i,p}}\cdot v_{j_i-1}(t).$$

$v_k(x_i)=0$ implies directly $C_{i,p}=0$, and therefore $v_k\equiv v_p\equiv 0$ on $[\alpha,x_{i+1}]$. If $v_k(x_i)>0$, we especially have $C_{i,k}>0$.
For all $t\in[x_i,x_{i+1}]$ follows

$$v_p(t)=\frac{C_{i,p}}{C_{i,k}}\cdot v_k(t),$$

and, in particular, for x_{i+1}

$$v_p(x_{i+1}) = \frac{C_{i,p}}{C_{i,k}} \cdot v_k(x_{i+1})$$

$$= \frac{C_{i+1,p}}{C_{i+1,k}} \cdot v_k(x_{i+1}).$$

Since $v_k(x_i) > 0$, we have $v_k(x_{i+1}) > 0$, thus

$$\frac{C_{i,p}}{C_{i,k}} = \frac{C_{i+1,p}}{C_{i+1,k}}.$$

So we finally get

$$v_p \equiv \frac{C_{i+1,p}}{C_{i+1,k}} \cdot v_k$$

on the interval $[\alpha, \beta]$. Q.E.D.

This completes the proof of Theorem 1.

To prove Theorem 2 we need the following results:

LEMMA 8. *Let $c, d \in M$ and let $1, f_1, ..., f_n \in F(M)$ be a weak Markov system with Property* (E). *If $f_{1|[c,d] \cap M} \equiv \text{const}$, then $f_{|[c,d] \cap M} \equiv \text{const}$ for every $f \in U_n$.*

Proof. For $n \leqslant 1$ the statement is trivial.

$n-1 \Rightarrow n$: By Condition (E1) there exists a weak M^+ system $1, g_1, ..., g_n \in F(M)$ with $\text{lin}\{1, ..., g_j\} = \text{lin}\{1, ..., f_j\}$ for every $j \in \{1, ..., n\}$.

By the induction hypothesis every $g \in U_{n-1}$ is constant on $[c, d] \cap M$.

As U_n is a weak Tchebycheff space, there exists $\tilde{c}, \tilde{d} \in M$ with $\tilde{c} \leqslant c < d \leqslant \tilde{d}$, such that $1, f_1, ..., f_n$ are linearly independent on $[\tilde{c}, \infty) \cap M$ as well as on $(-\infty, \tilde{d}] \cap M$.

Now let $1, u_1, ..., u_n \in F(M)$ with Property (E2) on the set $[\tilde{c}, \infty) \cap M$ and let $1, v_1, ..., v_n \in F(M)$ with Property (E3) on $(-\infty, d] \cap M$.

Let $\text{card}(M \cap [c, d]) \geqslant 2$. Thus, for all $(t_1, t_2) \in \Delta_2(M \cap [c, d])$

$$\begin{vmatrix} 1 & 1 \\ (-1)^{n-1} v_n(t_1) & (-1)^{n-1} v_n(t_2) \end{vmatrix} = (-1)^{n-1} (v_n(t_2) - v_n(t_1))$$

$$= (-1)^{n-1} (g_n(t_2) - g_n(t_1))$$

$$\geqslant 0$$

holds, because of $1, (-1)^{-1} v_n$ is a weak M^+ system on the set $(-\infty, \tilde{d}] \cap M$, and $v_n = g_n + g$ with $g \in \text{lin}\{1, g_1, ..., g_{n-1}\}$.

First let us assume that there exists a point $\tilde{t} \in M$, $\tilde{t} < c$ with $f_1(\tilde{t}) \neq f_1(c)$. Applying Condition (E3) we get

$$\begin{aligned}
(-1)^{n-2} \begin{vmatrix} 1 & 1 & 1 \\ v_1(\tilde{t}) & v_1(t_1) & v_1(t_2) \\ v_n(\tilde{t}) & v_n(t_1) & v_n(t_2) \end{vmatrix}
&= (-1)^{n-2} \begin{vmatrix} 1 & 1 & 1 \\ v_1(\tilde{t}) - v_1(t_1) & 0 & 0 \\ v_n(\tilde{t}) & v_n(t_1) & v_n(t_2) \end{vmatrix} \\
&= (-1)^{n-2} (g_1(t_1) - g_1(\tilde{t}))(g_n(t_2) - g_n(t_1)) \\
&\geqslant 0.
\end{aligned}$$

So $g_1(t_1) - g_1(\tilde{t}) > 0$ implies $g_n(t_1) = g_n(t_2)$ for all $(t_1, t_2) \in \Delta_2([c, d] \cap M)$.

If $f_1 \equiv \text{const}$ on $(-\infty, d] \cap M$, there exists a point $\tilde{t} \in M$, $\tilde{t} > d$ with $f_1(d) \neq f_1(\tilde{t})$.

Using Condition (E2) we have

$$\begin{vmatrix} 1 & 1 \\ u_n(t_1) & u_n(t_2) \end{vmatrix} \geqslant g_n(t_2) - g_n(t_1) \geqslant 0,$$

$$\begin{vmatrix} 1 & 1 & 1 \\ u_1(t_1) & u_1(t_2) & u_1(\tilde{t}) \\ u_n(t_1) & u_n(t_2) & u_n(\tilde{t}) \end{vmatrix} = (g_1(t_1) - g_1(\tilde{t}))(g_n(t_2) - g_n(t_1)) \geqslant 0,$$

and $g_1(t_1) - g_1(\tilde{t}) < 0$. Thus, $g_n(t_2) = g_n(t_1)$ holds for all $(t_1, t_2) \in \Delta_2([c, d] \cap M)$, and the statement readily follows. Q.E.D.

DEFINITION. Let $f, g \in F(M)$. Then g is called

(a) C-bounded on M, if g is bounded on $[a, b] \cap M$ for every $a, b \in M$;

(b) Lipschitz-bounded with respect to f, if for every $a, b \in M$ there exists $K > 0$, such that

$$|g(x) - g(y)| \leqslant K\,|f(x) - f(y)| \qquad \text{for} \quad x, y \in [a, b] \cap M.$$

A weak Markov system $1, f_1, ..., f_n \in F(M)$ is called Lipschitz-bounded with respect to f_1 (C-bounded), if all functions $f_1, ..., f_n$ are Lipschitz-bounded with respect to f_1 (C-bounded).

In [9] Zalik proved C-boundedness for weak Markov systems with the Properties (E) and (I).

LEMMA 9. *Every normalized weak Markov system* $1, f_1, ..., f_n \in F(M)$ *with Property* (E) *is C-bounded.*

Proof. Obviously, it is sufficient to show C-boundedness for the system $1, g_1, ..., g_n \in F(M)$, given by Condition (E1).

For $n \leqslant 1$ the statement is trivial.

$n-1 \Rightarrow n$: Let us suppose there are c, $d \in M$ such that g_n is unbounded on the set $[c, d] \cap M$. Therefore, the function $g_n \in U_n$ possesses at least one pole $p \in [c, d] \cap \bar{M}$.

So there is a sequence $(t_k)_{k=0}^{\infty}$ in the set $[c, d] \cap M$ converging to p with $\lim_{k \to \infty} | g_n(t_k)| = \infty$. Without loss of generality let $t_0 > t_k$ for every $k \geqslant 1$.

Moreover, there is a point $\tilde{c} \in (-\infty, c] \cap M$, such that $1, g_1, ..., g_n$ are linearly independent on the set $[\tilde{c}, \infty) \cap M$.

By the induction hypothesis we have $u_n = g_n + g$ with $g \in U_{n-1}$, which is bounded on $[c, d] \cap M$. Condition (E2) implies that the sets $\{u_n\}$ and $\{1, u_n\}$ form weak M^+ systems on $[\tilde{c}, \infty) \cap M$. Thus, for each $k \geqslant 1$ there follows

$$u_n(t_k) = g_n(t_k) + g(t_k) \geqslant 0$$

and

$$u_n(t_0) - u_n(t_k) = -g_n(t_k) + (u_n(t_0) - g(t_k)) \geqslant 0.$$

Therefore, the unboundedness of the sequence $(g_n(t_k))_{k=0}^{\infty}$ leads to a contradiction. Q.E.D.

LEMMA 10. *Every normalized weak Markov system* $1, f_1, ..., f_n \in F(M)$ *with Property* (E) *is Lipschitz-bounded with respect to* f_1.

Proof. We are going to prove the statement for the weak M^+ system $1, g_1, ..., g_n \in F(M)$, given by Condition (E1).

If $n \leqslant 1$, the statement is obvious.

$n-1 \Rightarrow n$: Let $(c, d) \in \Delta_2(M)$ be fixed. There are $\tilde{c}$, $\tilde{d} \in M$ with $\tilde{c} \leqslant c < d \leqslant \tilde{d}$ and

$$\dim U_{n|[\tilde{c}, \infty) \cap M} = \dim U_{n|(-\infty, \tilde{d}] \cap M} = n+1.$$

Moreover, let us assume that $1, v_1, ..., v_n \in F(M)$ fulfill Condition (E3) on $(-\infty, \tilde{d}] \cap M$.

Case 1. There are $(\tilde{t}_0, \tilde{t}_1, \tilde{t}_2) \in \Delta_3([c, d] \cap M)$ with $g_1(\tilde{t}_0) < g_1(\tilde{t}_1) < g_1(\tilde{t}_2)$. By Condition (E3) the sets

$\{1, v_1\}$, $\{1, (-1)^{n-1} v_n\}$, $\{-v_1, (-1)^{n-1} v_n\}$, and $\{1, v_1, (-1)^{n-2} v_n\}$

form weak M^+ systems on $(-\infty, \tilde{d}] \cap M$. Therefore v_1 and $(-1)^{n-1} v_n$ are increasing on $(-\infty, \tilde{d}] \cap M$, and

$$(-1)^n \begin{vmatrix} v_1(x) & v_1(y) \\ v_n(x) & v_n(y) \end{vmatrix} = (-1)^n (v_n(y)\, v_1(x) - v_n(x)\, v_1(y)) \geqslant 0$$

for all $(x, y) \in \Delta_2(M \cap (-\infty, \tilde{d}])$.

Now let $(t_0, t_1, t_2) \in \Delta_3(M \cap (-\infty, \tilde{d}])$ be fixed, such that $g_1(t_0) < g_1(t_1) < g_1(t_2)$. Then

$$\begin{aligned}(-1)^{n-2} &\begin{vmatrix} 1 & 1 & 1 \\ v_1(t_0) & v_1(t_1) & v_1(t_2) \\ v_n(t_0) & v_n(t_1) & v_n(t_2) \end{vmatrix} \\ &= (-1)^{n-2} [(v_n(t_1)\, v_1(t_0) - v_n(t_0)\, v_1(t_1)) \\ &\quad - v_1(t_2)(v_n(t_1) - v_n(t_0)) + v_n(t_2)(v_1(t_1) - v_1(t_0))] \\ &= (-1)^{n-2} [(v_n(t_1) - v_n(t_0))(v_1(t_0) - v_1(t_2)) \\ &\quad + (v_n(t_2) - v_n(t_0))(v_1(t_1) - v_1(t_0))] \\ &=: D_I \geqslant 0.\end{aligned}$$

By a simple calculation one shows

$$\begin{aligned}&\frac{D_I}{(v_1(t_2) - v_1(t_0))(v_1(t_1) - v_1(t_0))} \\ &\qquad = (-1)^{n-2} \left(\frac{v_n(t_2) - v_n(t_0)}{v_1(t_2) - v_1(t_0)} - \frac{v_n(t_1) - v_n(t_0)}{v_1(t_1) - v_1(t_0)} \right) \geqslant 0.\end{aligned}$$

Let $t_0 \in (-\infty, d) \cap M$ be fixed. Then

$$\varphi_{t_0}(x) := (-1)^{n-2} \frac{v_n(x) - v_n(t_0)}{g_1(x) - g_1(t_0)}$$

is well defined on the set $M_{t_0} := \{ t \in (t_0, \infty) \cap M \mid g_1(t_0) < g_1(t) \}$.

As the functions v_1 and $(-1)^{n-1} v_n$ are increasing on $(-\infty, \tilde{d}] \cap M$, φ_{t_0} is nonpositive, increasing, and bounded from above.

A similar computation of the determinant D_I gives

$$\begin{aligned}&\frac{D_I}{(v_1(t_2) - v_1(t_0))(v_1(t_2) - v_1(t_0))} \\ &\qquad = (-1)^{n-2} \left(\frac{v_n(t_1) - v_n(t_2)}{v_1(t_1) - v_1(t_2)} - \frac{v_n(t_0) - v_n(t_2)}{v_1(t_0) - v_1(t_2)} \right) \geqslant 0.\end{aligned}$$

For fixed $t_2 \in (-\infty, d] \cap M$ the function

$$\varphi_{t_2}(x) = (-1)^{n-2} \frac{v_n(x) - v_n(t_2)}{g_1(x) - g_1(t_2)}$$

is increasing, nonpositive on $M_{t_2} := \{t \in (-\infty, t_2) \cap M \mid g_1(t) < g_1(t_2)\}$, and therefore bounded from above.

Applying the induction hypothesis to $g_n = v_n + g$, $g \in U_{n-1}$, the Lipschitz-boundedness of g_n directly follows from the fact that φ_{t_1} and φ_{t_2} are bounded from above.

Case 2. If $g_1([c, d] \cap M)$ consists of no more than two points, the proof of the statement follows by Lemma 8. Q.E.D.

Throughout the following considerations on relative derivatives we can assume:

1. $I = (a, b)$ an open and bounded interval
2. $1, f_1, ..., f_n \in C(I)$ a normalized weak Markov system.

These assumptions mean no loss of generality, because in [8] Zalik proved the following embedding property of weak Markov systems:

Every C-bounded normalized weak M^+ system $1, f_1, ..., f_n \in F(M)$ is embeddable in a normalized weak M^+ system $1, g_1, ..., g_n \in C(I)$, where I is an open-bounded interval, i.e., there is $c \in M$ and a strictly increasing function $h: M \to I$ with $h(c) = c$, such that $g_j(h(t)) = f_j(t)$ for every $j \in \{0, ..., n\}$ and every $t \in M$. Examining the proof one sees that if $1, f_1, ..., f_n \in F(M)$ has Property (E) this also holds for $1, g_1, ..., g_n \in C(I)$ (see Theorem 3 in [9]).

Examining the proof one sees that if $1, f_1, ..., f_n \in F(M)$ has Property (E) this also holds for $1, g_1, ..., g_n \in C(I)$ (see Theorem 3 in [9]).

DEFINITION. Let $f, g \in C(I)$, f monotone and nonconstant, and for $\alpha \in I$ let

$$R_\alpha := \{x \in (\alpha, b) \mid f(\alpha) \neq f(x)\}, \qquad L_\alpha := \{x \in (a, \alpha) \mid f(x) \neq f(\alpha)\},$$

$$r_\alpha := \inf R_\alpha, \qquad l_\alpha := \sup L_\alpha.$$

Moreover, let

$$I_R := \{x \in I \mid R_x \neq \varnothing\}, \qquad I_L := \{x \in I \mid L_x \neq \varnothing\}.$$

Then the right and left relative derivatives of g with respect to f are defined by

$$D_+ g(\alpha) = \lim_{t \to r_{\alpha+}} \frac{g(t)-g(\alpha)}{f(t)-f(\alpha)}, \qquad \alpha \in I_R$$

and

$$D_- g(\alpha) = \lim_{t \to l_{\alpha-}} \frac{g(t)-g(\alpha)}{f(t)-f(\alpha)}, \qquad \alpha \in I_L.$$

The concept of relative differentiation in normalized weak Markov spaces was introduced by Zielke in [11].

To prove Theorem 2 we need the following result, which may be of some independent interest:

THEOREM 3. *If* $1, f_1, ..., f_n \in C(I)$ *is a weak Markov system with Property* (E), *then*

$$D_+ f_1, ..., D_+ f_n \in F(I_R)$$

and

$$D_- f_1, ..., D_- f_n \in F(I_L)$$

are normalized weak Markov systems with Property (E).

LEMMA 11. *Let* $1, f_1, ..., f_n \in C(I)$ *be Lipschitz-bounded with respect to* f_1. *Then for every* $g \in U_n$

(a) $D_+ g(t) \in \mathbb{R}$ *for all* $t \in I_R$;

(b) $D_- g(t) \in \mathbb{R}$ *for all* $t \in I_L$.

The proof of Lemma 11 is completely analogous to the last part of the proof of Lemma 11.3(a) in [10], and will therefore be omitted.

LEMMA 12. *Let* $g \in U_n$ *be Lipschitz-bounded with respect to* f_1. *Then*

(a) $D_+ g(t) = 0$ *for all* $t \in (c, d) \subset I_R$ *implies* $g \equiv \text{const}$ *on* (c, d);

(b) $D_- g(t) = 0$ *for all* $t \in (c, d) \subset I_L$ *implies* $g \equiv \text{const}$ *on* (c, d).

Proof. Without loss of generality we may assume that f_1 is increasing. At first, let g be increasing, too.

Fix $\varepsilon > 0$ and let $x_0 \in (c, d)$. Because of $D_+ g \equiv 0$ on (c, d) we have for $x > r_{x_0}$

$$0 < \frac{g(x)-g(x_0)}{f(x)-f(x_0)} < \varepsilon,$$

if the distance $|x - r_{x_0}|$ is sufficiently small.

The above estimate implies

$$\varepsilon f(x_0) - g(x_0) < \varepsilon f(x) - g(x).$$

By Riesz's lemma (see, e.g., [3, p. 319]), for each closed interval $[\gamma, \delta] \subset (c, d)$ there follows

$$\varepsilon f(\gamma) - g(\gamma) \leqslant \varepsilon f(\delta) - g(\delta),$$

and therefore

$$g(\delta) - g(\gamma) \leqslant \varepsilon(f(\delta) - f(\gamma)).$$

Since $\varepsilon > 0$ was arbitrary, $g([\gamma, \delta]) = [g(\gamma), g(\delta)]$ is a degenerated interval, thus $g \equiv \text{const}$ on (c, d).

Now, let $g \in U_n$ be arbitrary. By Lemma 3 there exists a natural number $k \leqslant n+1$, and points $p_0, ..., p_k$ with $c = p_0 < \cdots < p_k = d$, such that g is monotone on each interval (p_j, p_{j+1}), $j \in \{0, ..., k-1\}$.

Thus, $g \equiv \text{const}$ on every interval (p_j, p_{j+1}). Since $D_+ g(p_j) = 0$, $j \in \{1, ..., k-1\}$, we get $g \equiv \text{const}$ on (c, d).

The proof of part (b) is completely analogous to the proof of part (a) and will be omitted. Q.E.D.

Proof of Theorem 3. One easily sees from Lemmas 10 and 11 that $D_+ : U_n \to F(I_R)$ is a well-defined linear operator.

Clearly, kern D_+ contains U_0, so $D_+ U_n$ is a subspace with $\dim D_+ U_n \leqslant n$. Applying Lemma 12 it follows $U_0 = \text{kern}\, D_+$, and therefore $\dim D_+ U_n = \dim U_n - \dim(\text{kern}\, D_+) = n$. Proceeding as in [10, Lemma 11.3(b)] we conclude that $D_+ f_1, ..., D_+ f_n \in F(I_R)$ is a normalized weak Markov system.

By Condition (E1) there exists a normalized weak M^+ system 1, $g_1, ..., g_n \in C(I)$, such that for each $j \in \{1, ..., n\}$

$$\text{lin}\{1, ..., f_j\} = \text{lin}\{1, ..., g_j\}.$$

We show that $D_+ g_1, ..., D_+ g_n \in F(I_R)$ is a normalized weak M^+ system, if f_1 is increasing; if f_1 is decreasing, then $-D_+ g_1, ..., -D_+ g_n \in F(I_R)$ is a normalized weak M^+ system:

Let f_1 be increasing, $k \in \{1, ..., n\}$, $(t_1, ..., t_k) \in \Delta_k(I_R)$ and $\varphi \in D_+ U_k$ with

$$\varphi = \alpha D_+ g_k + \tilde{\varphi}, \qquad \alpha > 0, \tilde{\varphi} \in D_+ U_{k-1}.$$

Suppose that

$$(-1)^{k-i} \varphi(t_i) < 0 \qquad \text{for} \quad i = 1, ..., k.$$

Then there are functions $g \in U_k$ and $\tilde{g} \in U_{k-1}$ such that $\varphi = D_+ g$, $g = \alpha g_k + \tilde{g}$, and $\tilde{\varphi} = D_+ \tilde{g}$.

Since $(-1)^{k-i} \varphi(t_i) < 0$ for each $i \in \{1, ..., k\}$, there exists $(u_1, ..., u_k) \in \Delta_k(I)$ with $u_k \in (t_k, b)$ and $u_i \in (t_i, t_{i+1})$ for any $i \in \{1, ..., k-1\}$, such that

$$(-1)^{k-i} \frac{g(u_i) - g(t_i)}{f_1(u_i) - f_1(t_i)} < 0 \qquad i = 1, ..., k.$$

As f_1 is increasing we have

$$(-1)^{k-i} (g(u_i) - g(t_i)) < 0 \qquad i = 1, ..., k.$$

Consequently $(t_1, u_1, ..., t_k, u_k) \in \Delta_{2k}(I)$ contains a negatively oriented oscillation of $g = \alpha g_k + \tilde{g} \in U_k$, in contradiction to Lemma 4.

If f_1 is decreasing, the proof is completely analogous.

Conditions (E2) and (E3) can be shown by analogous arguments.

Q.E.D.

Note, that the oscillation Lemma 4 for normalized weak M^+ systems was essential to prove Property (E) for the relative derivatives.

Proof of Theorem 2. Let $1, f_1, ..., f_n$ be a weak Markov system with Property (E).

For $n \leqslant 1$ the statement is trivial.

$n-1 \Rightarrow n$: By the embedding property of weak Markov systems $1, f_1, ..., f_n$ is embeddable in a weak Markov system $1, z_1, ..., z_n \in C(I)$, $I := (a, b)$ open and bounded, i.e., there is $c \in M$, and a strictly increasing function $h_1 : M \to I$ with $h_1(c) = c$, such that $f_j(x) = z_j(h_1(x))$ for every $j \in \{1, ..., n\}$ and for every $x \in M$; $1, z_1, ..., z_n$ has Property (E).

From Theorem 3 follows that the left and right relative derivatives of $1, z_1, ..., z_n$ are nomalized weak Markov systems with Property (E).

Now let I_R and I_L be defined as above. If there is $\alpha \in I$, such that $z_1 \equiv z_1(\alpha)$ on $[\alpha, b)$, let us define

$$b_1 := \inf\{x \in I \mid z_1(x) = z_1(\alpha)\}$$

and $b_1 := b$, if there is no such α.

If $b_1 < b$, we have $\sup I_R = b_1 \in I_R$.

By Lemma 9, $1 = D_+ z_1, ..., D_+ z_n \in F(I_R)$ is C-bounded. Thus there is a normalized weak Markov system $1, \varphi_1, ..., \varphi_n \in F(I)$ such that for each $j \in \{2, ..., n\}$

$$\varphi_j|_{I_R} \equiv D_+ z_j$$

and, if $b_1 < b$

$$\varphi(x) = D_+ z_j(b_1), \qquad x \in [b_1, b).$$

Obviously, $1=\varphi_1, ..., \varphi_n \in F(I)$ has Property (E), and, using the induction hypothesis, it is representable. So there is $\tilde{c} \in I$, a strictly increasing function $h_2 : I \to \mathbb{R}$ with $h_2(\tilde{c}) = \tilde{c}$, and increasing functions $w_2, ..., w_n \in C(K(h_2(I)))$ with $w_2(\tilde{c}) = \cdots = w_n(\tilde{c}) = 0$, such that for every $j \in \{2, ..., n\}$ and for every $x \in I$

$$\varphi_j(x) = \int_{\tilde{c}}^{h(x)} \cdots \int_{\tilde{c}}^{t_{j-1}} dw_j(t_j) \cdots dw_2(t_2).$$

Now, let us define ϕ_j on the convex hull of $h_2(I)$ by

$$\phi_j(t) = \int_{\tilde{c}}^{t} \cdots \int_{\tilde{c}}^{t_{j-1}} dw_j(t_j) \cdots dw_2(t_2), \qquad j = 2, ..., n.$$

Without loss of generality we may choose $\tilde{c} = c$.

Let w_1 be defined by $w_1(x) = z_1(h_2^{-1}(x))$, $x \in h_2(I)$, and on the convex hull of $h_2(I)$ by linear interpolation in the same way as in the proof of Theorem 3 in [11].

Setting $h = h_2 \circ h_1$, then for $x \in M$ and $j \in \{1, ..., n\}$ we get

$$g_j(x) := f_j(x) - f_j(c) = \int_c^{h(x)} \phi_j(s)\, dw_1(s)$$

an integral representation of $1, f_1, ..., f_n \in F(M)$.

Now, let $1, f_1, ..., f_n \in F(M)$ be representable. Then there is a basis $1, g_1, ..., g_n \in F(M)$ of U_n defined by (I). Obviously, it is sufficient to show Property (E) for the corresponding system $1, v_1, ..., v_n \in C(J)$ defined by (I1).

By Lemma 5, $1, v_1, ..., v_n \in C(J)$ is a weak Markov system.

Proceeding by induction over n, we will prove

(1) $1, v_1, ..., v_n \in C(J)$ is a normalized weak M^+ system.

Proof of (1). For $n=0$ the statement is trivial.

$n-1 \Rightarrow n$: Let $v \in \operatorname{lin}\{1, ..., v_n\}$, say

$$v = \sum_{i=0}^{n} \alpha_i v_i, \qquad \text{with } \alpha_n > 0, \alpha_i \in \mathbb{R} \qquad \text{for} \quad i = 0, ..., n-1$$

and let us suppose that v has a negatively oriented alternation of length $n+1$ in $(t_0, ..., t_n) \in \Delta_{n+1}(J)$, i.e.,

$$(-1)^{n-j}\, v(t_j) < 0, \qquad j = 0, ..., n.$$

Then, for every $j \in \{1, ..., n\}$;

$$0 > (-1)^{n-j}\,(v(t_j) - v(t_{j-1})) = (-1)^{n-j} \left(\sum_{i=1}^{n} \alpha_i v_i(t_j) - \sum_{i=1}^{n} \alpha_i v_i(t_{j-1}) \right).$$

Clearly,

$$(v-\alpha_0)(t)=\left(\sum_{i=1}^{n}\alpha_i v_i\right)(t)=\int_c^t u(s)\,dw_1(s)$$

with

$$u=\alpha_1+\sum_{i=2}^{n}\alpha_i u_i,$$

where $u_2, ..., u_n$ are defined by (I2); note that, by the induction hypothesis, $1, u_2, ..., u_n$ is a weak M^+ system. Therefore, for every $j \in \{1, ..., n\}$, there exists $\zeta_j \in [t_{j-1}, t_j]$, such that

$$0>(-1)^{n-j}\int_{t_{j-1}}^{t_j} u(s)\,dw_1(s)=(-1)^{n-j}\,u(\zeta_j)\underbrace{(w_1(t_j)-w_1(t_{j-1}))}_{\geqslant 0}.$$

But then, u has in $(\zeta_1, ..., \zeta_n) \in \Delta_n(J)$ a negatively oriented alternation of length n, in contradiction to Lemma 2.

It is easy to see that it is sufficient to prove Condition (E2) on $[c, \infty) \cap J$ and Condition (E3) on $(-\infty, c] \cap J$.

(2) If $1, v_1, ..., v_n$ is linearly independent on $[c, \infty) \cap J$, and $(k(l))_{l=0}^m$ and arbitrarily fixed subsequence of $\{0, ..., n\}$, then $v_{k(0)}, ..., v_{k(m)}$ is a weak M^+ system on $[c, \infty) \cap J$.

Proof of (2). We distinguish two subcases.

Case 1. $k(0)=0$.
For $m=0$ the statement is obvious.
$m-1 \Rightarrow m$: Then, for all $t \in [c, \infty) \cap J$,

$$v_{k(1)}(t)=\int_c^t dv_{k(1)}(t_{k(1)})$$

and, for all $i \in \{2, ..., m\}$,

$$v_{k(i)}(t)=\int_c^t\int_c^{t_{k(1)}}\cdots\int_c^{t_{k(i)-1}} dw_{k(i)}(t_{k(i)})\cdots dw_{k(1)+1}(t_{k(1)+1})\,dv_{k(1)}(t_{k(1)}).$$

On the set $[c, \infty) \cap J$, $v_{k(1)}$ is increasing and nonnegative. Now, proceeding as in the proof of Lemma 13.2 in [10], and following the arguments used in (1) one gets: $1, v_{k(1)}, ..., v_{k(m)}$ is a weak M^+ system on $[c, \infty) \cap J$.

If $k(0)>0$, these arguments are not applicable. But in that case we have $v(c)=0$ for every $v \in \operatorname{lin}\{v_{k(0)}, ..., v_{k(m)}\}$.

Case 2. $k(0)>0$.

If $m=0$, then the statement follows by the fact that $v_{k(0)}$ is increasing on $[c, \infty] \cap J$.

$m-1 \Rightarrow m$: Let us suppose that there are $v \in \operatorname{lin}\{v_{k(0)}, ..., v_{k(m)}\}$ and $(t_0, ..., t_{m+1}) \in \Delta_{m+2}([c, \infty) \cap J)$, such that

$$(-1)^{m+1-j} v(t_j) < 0, \qquad j=0, ..., m+1.$$

$v(c)=0$ implies $c<t_0$. Setting $t_{-1} := c$, it follows that

$$(-1)^{m+1-j} \int_{t_{j-1}}^{t_j} \tilde{v}(s)\, dv_{k(0)}(s) < 0, \qquad j=0, ..., m+1$$

with $\tilde{v} \in \operatorname{lin}\{\tilde{v}_{k(0)}, ..., \tilde{v}_{k(m)}\}$, where

$$\tilde{v}_{k(0)}(t)=1$$

$$\tilde{v}_{k(i)}(t)=\int_c^t \cdots \int_c^{t_{k(i)-1}} dw_{k(i)}(t_{k(i)}) \cdots dw_{k(0)+1}(t_{k(0)+1})$$

for $i \in \{1, ..., m\}$.

But then, proceeding completely analogously to the proof of [10, Lemma 13.2] $\tilde{v}$ would have a strong alternation of length $m+2$ in $[c, \infty) \cap J$, a contradiction.

Moreover, using the fact that $v(c)=0$ for every $v \in \operatorname{lin}\{v_{k(0)}, ..., v_{k(m)}\}$, and, following the arguments of (1) one gets: $1, \tilde{v}_{k(1)}, ..., \tilde{v}_{k(m)}$ is a weak M^+ system on the set $[c, \infty) \cap J$.

The proof of Condition (E3) is completely analogous to the proof of Condition (E2). Q.E.D.

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